A General Theory of Rank Testing

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  2. It is not clear what relationships exist between the various rank testing statistics.
  3. It does not take full advantage of the numerical analysis literature.
  4. There is no fixed–b theory of rank testing (Kiefer et. al. (2000), Vogelsang (2001), Kiefer & Vogelsang (2002a,b, 2005)).

The Main Contribution

- The general structure of every rank testing statistic is:

$$T^{\theta}\tau\left(\underbrace{\{x_1,\ldots,x_T\}}_{\text{data}},\ \underbrace{P_{\widehat{N}_r},\quad P_{\widehat{M}_r}}_{\text{null space estimators}}\right).$$

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- This is termed the **plug–in principle** for rank testing statistics.

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- Consider the regression model

$$y_t = \underset{n\times m}{B} x_t + \varepsilon_t, \qquad t = 1, \ldots, T,$$

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and the global alternative

$$H_1(r) : B = B^*, \qquad \operatorname{rank}(B^*) > r.$$

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$$F = T\text{vec}'(P_{\widehat{N}_r}\widehat{B}P_{\widehat{M}_r})\{(P_{\widehat{M}_r} \otimes P_{\widehat{N}_r})\widehat{\Omega}(P_{\widehat{M}_r} \otimes P_{\widehat{N}_r})\}^{\dagger}\text{vec}(P_{\widehat{N}_r}\widehat{B}P_{\widehat{M}_r}).$$

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- The Johansen (1988), Cragg & Donald (1996,1997), Robin & Smith (2000), Kleibergen and Paap (2006), and Donald, Fortuna, & Pipiras (2007) statistics are all of this form.
- They differ only in their constructions of $P_{\widehat{N}_r}$ and $P_{\widehat{M}_r}$.

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- For symmetric positive definite B, Donald, Fortuna, & Pipiras (2007) have proposed:

$$t = \frac{\sqrt{T}\mathrm{tr}(P_{\widehat{M_r}}\widehat{B}P_{\widehat{M_r}})}{\sqrt{\mathrm{vec}'(I_m)(P_{\widehat{M_r}} \otimes P_{\widehat{M_r}})\widehat{\Omega}(P_{\widehat{M_r}} \otimes P_{\widehat{M_r}})\mathrm{vec}(I_m)}}.$$

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- All of the above (and many more) have the form $T^{\theta}\tau(\widehat{B},\widehat{\Omega},P_{\widehat{N}_r},P_{\widehat{M}_r}) = T^{\theta}\kappa(P_{\widehat{N}_r}\widehat{B}P_{\widehat{M}_r},(P_{\widehat{M}_r}\otimes P_{\widehat{N}_r})\widehat{\Omega}(P_{\widehat{M}_r}\otimes P_{\widehat{N}_r}))$.

Null Spaces Estimation: the Basic Idea

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- There are numerous algorithms in the literature that can identify the effective rank of a matrix (SVD, GSVD, WLRA, LU, QR, etc.).

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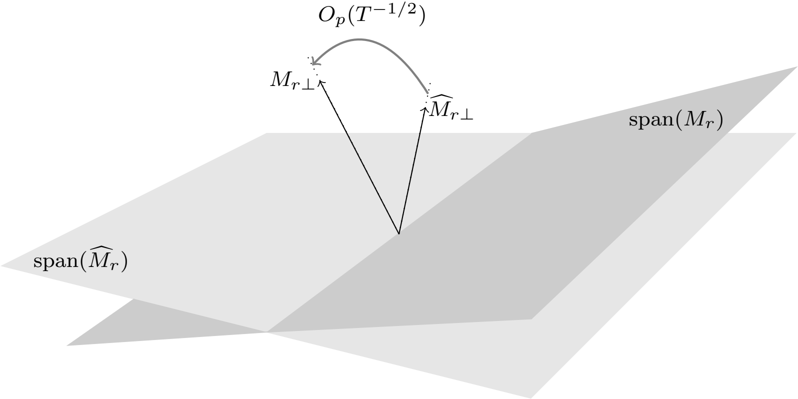
2. *If* $0 \leq i < r$ *then* $P_{\widehat{N}_i} \widehat{B} P_{\widehat{M}_i}$ *is bounded away from zero in probability.*
3. *If* $0 \leq i < r$ *and the RRA is continuous at* B^*, *then* $P_{\widehat{N}_i} \widehat{B} P_{\widehat{M}_i}$ *converges in probability.*

Null Space Estimation in General

Figure: Convergence of a Two Dimensional Null Space Estimator in $\mathbb{R}^3$.



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Similar assumptions are made for testing the rank of symmetric matrices.

The Feasible and Infeasible Statistics

Let $\widehat{N}_r$ and $\widehat{M_r}$ be null space estimators based on $\widehat{B}$. The **feasible** rank test statistic is:

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Let N_r and M_r span the null spaces of B^*. The **infeasible** rank test statistic is:

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2. Under $H_1(r)$, if the infeasible statistic has power, then so does the feasible statistic.

   It is said to satisfy the **strong plug–in principle** relative to the null spaces of B^* if additionally

3. Under $H_1(r)$, the feasible and infeasible statistics diverge at the same rate.

Testing Rank Hypotheses for General Matrices

Theorem

Under weak regularity conditions on τ*,* $T^{\theta}\tau(\widehat{B}, \widehat{\Omega}, P_{\widehat{N}_r}, P_{\widehat{M}_r})$ *satisfies the weak plug–in principle for rank test statistics. If, additionally, the RRA is continuous at* B^**, then the statistic satisfies the strong plug–in principle.*

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- All of the rank testing statistics of the literature (and many more) satisfy the weak plug–in principle.

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- The Cragg & Donald (1996,1997) and Kleibergen and Paap (2006) statistics are asymptotically equivalent. When $\widehat{\Omega}$ is a Kronecker product, we may add to the list the statistics of Anderson (1951) and Robin & Smith (2000).

Testing Rank Hypotheses for General Matrices

Theorem

Under weak regularity conditions on τ, $T^{\theta}\tau(\widehat{B}, \widehat{\Omega}, P_{\widehat{N}_r}, P_{\widehat{M}_r})$ satisfies the weak plug–in principle for rank test statistics. If, additionally, the RRA is continuous at B^*, then the statistic satisfies the strong plug–in principle.

- All of the rank testing statistics of the literature (and many more) satisfy the weak plug–in principle. When the underlying RRA is continuous at the population matrix (a feature we have conjectured to be generic for all RRAs), also satisfy the strong plug–in principle.
- The Cragg & Donald (1996,1997) and Kleibergen and Paap (2006) statistics are asymptotically equivalent. When $\widehat{\Omega}$ is a Kronecker product, we may add to the list the statistics of Anderson (1951) and Robin & Smith (2000).
- As there are no first–order differences between these statistics, we must look for either higher–order difference or Monte Carlo performance for guidance.

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$$\left(\sqrt{T}\mathrm{vec}(N_r'\widehat{B}M_r), (M_r \otimes N_r)'\widehat{\Omega}(M_r \otimes N_r)\right) \xrightarrow{d} (\xi_r, \Omega_r),$$

then we have

$$F \xrightarrow{d} \xi_r'\Omega_r^\dagger\xi_r \qquad\qquad t \xrightarrow{d} \frac{\mathrm{tr}(\mathrm{mat}(D_{m-r}\xi_r))}{(\mathrm{vec}'(I_{m-r})\Omega_r\mathrm{vec}(I_{m-r}))^{1/2}}.$$

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Cointegration Caveat I: Johansen (1988,1995)

Consider the following VAR(1)

$$\Delta y_t = B y_{t-1} + \varepsilon_t, \qquad t = 1, \ldots, T.$$

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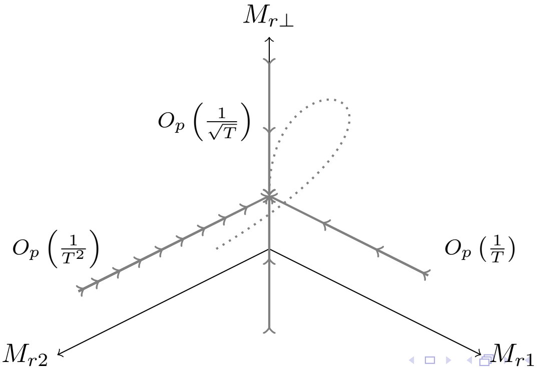
Figure: Convergence of the Columns of $\widehat{B}$.



Cointegration Caveat II: Nyblom & Harvey (2000)

Consider the multivariate local level model

$$y_t = x_t + \varepsilon_t \qquad x_t = x_{t-1} + u_t, \qquad t = 1, \ldots, T.$$

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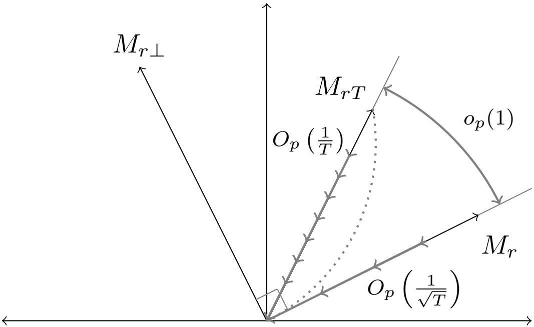

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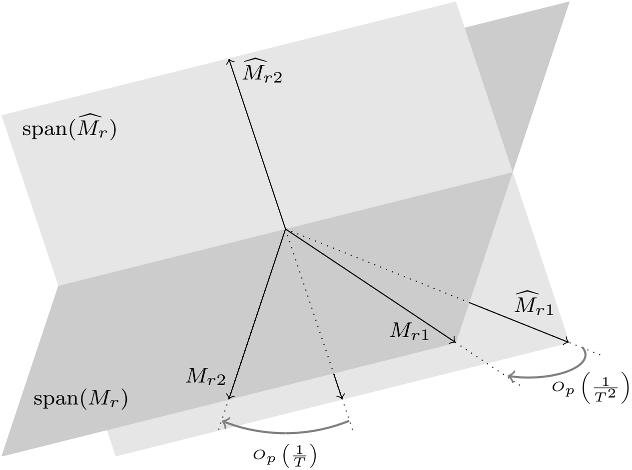
- The general plug–in principle is relative to a random sequence of matrices.
- The general theory nests the standard asymptotics case, as well as the polynomial regression and the vast majority of cointegration settings.

Null Space Estimation in Cointegration

Figure: Accelerated & Heterogeneous Rates of Null Space Convergence in $\mathbb{R}^3$.



Corollaries of the Plug–in Principle for Cointegration

- (Correct Specification). The limiting behaviour of all of the statistics in:

  Johansen (1988), Johansen (1991), Kleibergen & van Dijk (1994), Yang & Bewley (1996), Quintos (1998), Gonzalo & Pitarakis (1999), Lutkepohl & Saikkonen (1999), Kleibergen & Paap (2006), Avarucci & Velasco (2009), Cavaliere et al. (2010a),...

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  follow from Corollaries 3 and 4 of the paper.
- (Misspecification). The F statistics proposed by Johansen (1988), Kleibergen & van Dijk (1994), and Kleibergen & Paap (2006) have the exact same behaviour under the misspecification conditions of Caner (1998) (infinite variance shocks), Cavaliere et al. (2010b) (heteroskedastic shocks), and Aznar & Salvador (2002) and Cavaliere et al. (2014) (misspecified lag length).

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- Let the data be generated as

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y_t &= Bx_t + \varepsilon_t \\
\varepsilon_t &= 0.5\varepsilon_{t-1} + u_t \quad \text{(stationary)} \\
\{(x_t', u_t')'\} &\text{ i.i.d. } N(0, I_8) \\
B &= \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
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- PP plots are reported for F.

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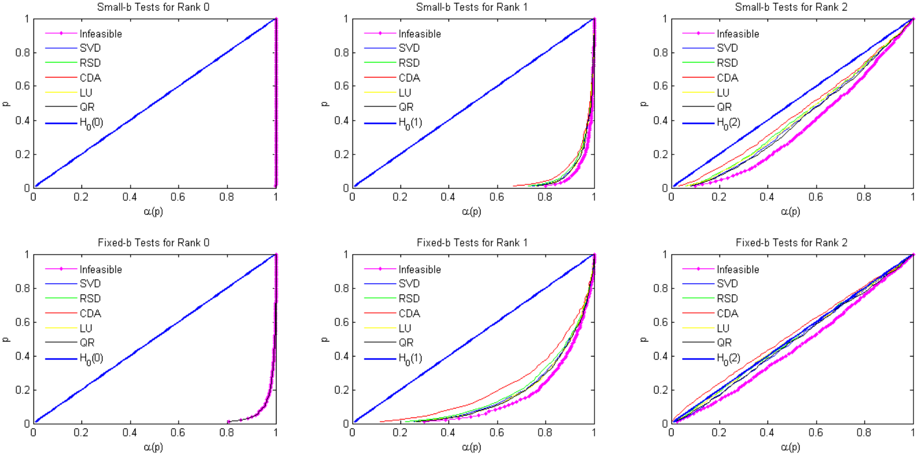
Figure: PP Plots for the F Statistic of Example 1.



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- Rank–1 RRAs are discontinuous at B. Therefore, we expect the different F statistics for rank–1 to diverge at different rates.

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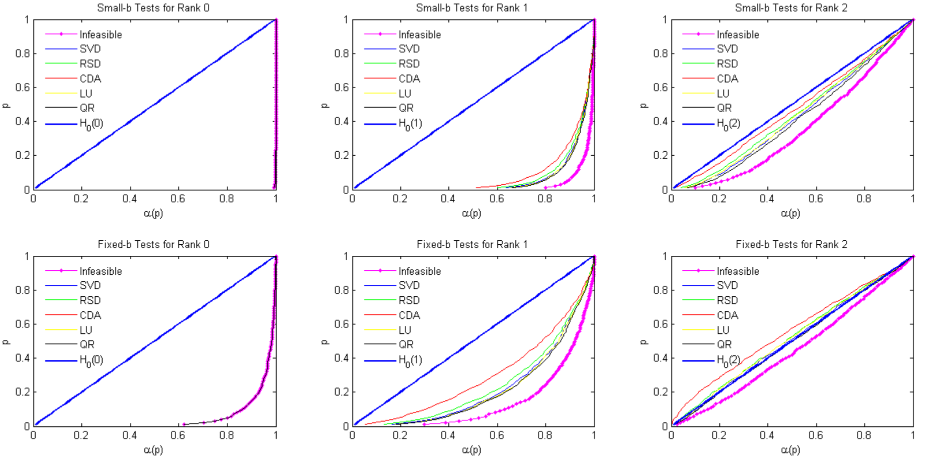

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- Future research to focus on high-dimensional data.